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IMAGE FORMATION IN CIRCULAR WAVEGUIDES AND OPTICAL FIBERS

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Abstract

The formation of images of Gaussian beams at certain distances along a circular metallic waveguide or optical fiber is studied. The phenomenon of Fourier imaging in which an essentially exact replica of the input beam is produced, occurs here as it does for slab waveguides. The Fresnel images produced in the circular guides, however, are not replicas, but are related to a certain transform of the original beam shape.

Introduction

It has been known for some time that a field distribution launched into certain multimode waveguides can reproduce itself with great accuracy at certain distances (known as Fourier image distances) from the input plane.[1-4] This property can be used in the design of devices such as couplers, switches and so forth. At certain other distances from the input plane, the parallel-plate or dielectric slab waveguide can also form Fresnel images--multiple and perhaps inverted replicas of the original field pattern reduced in amplitude and regularly spaced across the width of the guide.[2,4] These, too, have some potential for practical use, as in a signal splitter.

It is also known that circular metallic waveguides and step-index circular optical fibers can produce Fourier images. Nothing, however, seems to be known about the possible existence of Fresnel-type images on these structures. In this paper, we will outline a first attempt to study this problem.

Field Description by Modes

Consider a step-index circular fiber of radius a as shown in Fig. 1. The core refractive index is n_0, the cladding index is n_1. Modes of this fiber which are not too close to cutoff can be described[5] using the fields and propagation constants of the TM modes of a hollow metallic circular waveguide of slightly larger radius

$$b = a + k_0^{-1}(1 - n_1^2/n_0^2)^{-\frac{1}{2}} \tag{1}$$

where $k_0 = n_0\omega\sqrt{\mu_0\varepsilon_0}$ is the wavenumber in the core region, and a time dependence of $\exp(i\omega t)$ is assumed. The propagation constants for the two guides are identical in this approximation, and the function $E(\rho,z)$ which describes the longitudinal electric field of the equivalent hollow guide will also describe either of the transverse components of the electric field of the circular fiber. The following therefore applies equally to either type of waveguide.

Suppose that an azimuthally symmetric field distribution $E_0(\rho) = E(\rho,0)$ is present at the input plane $z=0$. Then only ϕ-independent modes are produced in the waveguide, and by well-known methods,[6] we can express the field at any point $z > 0$ along the guide as:

$$E(\rho,z) = \int_0^b \rho' E_0(\rho')K(\rho,\rho';z)d\rho' \tag{2}$$

where

$$K(\rho,\rho';z) = \frac{2}{b^2}\sum_{m=1}^{\infty} \frac{J_0(j_{om}\frac{\rho}{b})J_0(j_{om}\frac{\rho'}{b})}{J_1^2(j_{om})} e^{-i(k_0^2 - j_{om}^2/b^2)^{\frac{1}{2}}z} \tag{3}$$

Here J_n are Bessel functions and j_{om} is the $m^{\underline{th}}$ root of the Bessel function J_0.

Paraxial Approximation and Imaging

Assuming the waveguide to support a large number of modes above cutoff, most of these modes have propagation constants which will be well approximated by the paraxial approximation:

$$(k_0^2 - j_{om}^2/b^2)^{\frac{1}{2}} \simeq k_0 - j_{om}^2/2k_0b^2 \tag{4}$$

If we further invoke the asymptotic relation $j_{om} \sim (m-\frac{1}{4})\pi$, then at any multiple of the Fourier imaging distance

$$z_{11} = 8k_0b^2/\pi \tag{5}$$

we will have reproduced the original field pattern, with some phase shift only:

$$\begin{aligned} K(\rho,\rho';z_{11}) &\simeq e^{-ik_0z_{11}} \frac{2}{b^2}\sum_{m=1}^{\infty} \frac{J_0(j_{om}\frac{\rho}{b})J_0(j_{om}\frac{\rho'}{b})}{J_1^2(j_{om})} e^{i\pi 4(m-\frac{1}{4})^2} \\ &= e^{-ik_0z_{11}+i\pi/4} K(\rho,\rho';0) \end{aligned} \tag{6}$$

Even with the approximations we have made, the fidelity of these Fourier images to the original pattern is excellent.

To test the possibility of Fresnel images, let us inquire about the field distribution at $z = z_{11}/2$. Making the paraxial approximation and letting j_{om} be used interchangeably with its asymptotic form as before, we have

$$K(\rho,\rho';z_{11}/2) \simeq e^{-ik_0z_{11}+i\pi/8} \frac{2}{b^2}\sum_{m=1}^{\infty} \frac{J_0(j_{om}\frac{\rho}{b})J_0(j_{om}\frac{\rho'}{b})}{J_1^2(j_{om})}(-1)^m \tag{7}$$

Now, if ρ is not too close to zero, it is valid to replace $J_0(j_{om}\rho/b)$ by its asymptotic form. By suitable manipulation, we can find that the asymptotic form of $(-1)^m J_0(j_{om}\,\rho/b)$ is the same as that of a certain other combination of Bessel function terms whose coefficients are independent of m:

$$(-1)^m J_0(j_{om}\frac{\rho}{b}) \simeq -\left(\frac{b-\rho}{2\rho}\right)^{\frac{1}{2}} \left\{ J_0[j_{om}(1-\frac{\rho}{b})] + J_1[j_{om}(1-\frac{\rho}{b})] \right\} \tag{8}$$

Hence, (7) can be split into two parts

$$K(\rho,\rho';z_{11}/2) \simeq \left(\frac{b-\rho}{2\rho}\right)^{\frac{1}{2}} e^{-ik_0z_{11}-7\pi i/8}[K(b-\rho;\rho';0) + K_1(b-\rho,\rho')] \tag{9}$$

where

$$K_1(\rho,\rho') = \frac{2}{b^2}\sum_{m=1}^{\infty} \frac{J_1(j_{om}\frac{\rho}{b})J_o(j_{om}\frac{\rho'}{b})}{J_1^2(j_{om})} \quad (10)$$

From (2) and (9), we see that $E(\rho,z_{11}/2)$ resolves into a portion resembling the input field E_o (though "inverted" with respect to the center of the guide) and a certain transform of this input field. In fact, if $E_o(\rho)$ is given by

$$E_o(\rho) = \sum_{m=1}^{\infty} E_m J_o(j_{om}\frac{\rho}{b}) \quad (11)$$

in accordance with equations (2)-(3), then we can define a transform of this field by

$$E_1(\rho) = \int_0^b \rho' E_o(\rho')K_1(\rho,\rho')d\rho' = \sum_{m=1}^{\infty} E_m J_1(j_{om}\frac{\rho}{b}) \quad (12)$$

Here E_m are some constants depending on the detailed nature of the input field E_o. _If_ we can somehow compute $E_1(\rho)$ given $E_o(\rho)$, then we can write

$$E(\rho,z_{11}/2) \simeq \left(\frac{b-\rho}{2\rho}\right)^{\frac{1}{2}} e^{-ik_oz_{11}=7\pi i/8}[E_o(b-\rho) + E_1(b-\rho)] \quad (13)$$

This will be the "Fresnel image" formed at $z=z_{11}/2$.

When the input beam is Gaussian,

$$E_o(\rho) = e^{-\rho^2/2w_o^2} \quad (14)$$

then $E_1(\rho)$ can be given explicitly. For, since

$$E_m = \frac{2}{b^2J_1^2(j_{om})}\int_0^b \rho E_o(\rho)J_o(j_{om}\frac{\rho}{b})d\rho \quad (15)$$

then we can define an auxiliary function

$$f(\rho) = b\sum_{m=1}^{\infty}\frac{E_m}{j_{om}} J_o(j_{om}\frac{\rho}{b}) \quad (16)$$

such that $E_1(\rho) = -f'(\rho)$, and we can also say that

$$\frac{bE_m}{j_{om}} = \frac{2}{b^2J_1^2(j_{om})}\int_0^b \rho f(\rho)\, J_o(j_{om}\frac{\rho}{b})d\rho \quad (17)$$

From (15), then, we have

$$\frac{b}{j_{om}}\int_0^b \rho E_o(\rho)\, J_o(j_{om}\frac{\rho}{b})d\rho = \int_0^b \rho f(\rho)J_o(j_{om}\frac{\rho}{b})d\rho \quad (18)$$

Equation (18) is simply a statement of the equality of two Hankel transforms. If both $E_o(\rho)$ and $f(\rho)$ are essentially zero for $\rho \geq b$, then we can take inverse Hankel transforms of both sides of (18), to get

$$E_1(\rho) = -f'(\rho) = -\frac{d}{d\rho}\int_0^{\infty} \hat{E}_o(u)\, J_o(u\,)du \quad (19)$$

where

$$\hat{E}_o(u) = \int_0^{\infty} \rho E_o(\rho)J_o(u\rho)d\rho \quad (20)$$

Using these relationships, we can evaluate $E_1(\rho)$ for the Gaussian input field (14), and get

$$E_1(\rho) = \frac{\rho}{2w_o}\sqrt{\frac{\pi}{2}}\, e^{-\rho^2/4w_o^2}\left[I_o\left(\frac{\rho^2}{4w_o^2}\right) - I_1\left(\frac{\rho^2}{4w_o^2}\right)\right] \quad (21)$$

where I_o and I_1 are modified Bessel functions of the first kind.

In Fig. 2, we show a comparison of the prediction of eqn. (13) with that of the exact mode series. While the general shape is quite accurate, the curves differ by a constant factor over most of the range of ρ, and the approximate field actually blows up as $\rho \to 0$. The reason for this can be seen by returning to our central approximation (8). This is a nonuniform approximation; it blows up at $\rho = 0$ (as we have seen), and is not likely to be very accurate for ρ near b since it is based on asymptotic relations valid for $j_{om}(1-\rho/b)$ large. But for an input field originally concentrated near $\rho = 0$, our analysis has predicted that the important parts of the Fresnel image at $z=z_{11}/2$ are those near $\rho = b$, so eqn. (8) is an inadequate approximation for this purpose.

From an examination of Fig. 2, we speculate that the introduction of a constant factor into (8) and (13) may improve matters everywhere except very near $\rho=0$. Indeed, this turns out to be the case if we choose the factor $(2/\pi)^{\frac{1}{2}}$, making (8) now

$$(-1)^m J_o\ (j_{om}\frac{\rho}{b}) \simeq -\left(\frac{b-\rho}{\pi\rho}\right)^{\frac{1}{2}}\left\{J_o[j_{om}(1-\frac{\rho}{b})] + J_1[j_{om}(1-\frac{\rho}{b})]\right\} \quad (22)$$

Though (22) eventually breaks down for large enough m, for ρ near b it remains valid for a number of terms. At $\rho/b = 0.9$, for example, the left and right sides of (22) are as follows:

m = 1 : - .14 vs. -.21
m = 2 : - .19 vs. -.22
⋮
m = 5 : - .22 vs. -.20

So long as the input field excites a reasonably large number of modes significantly (so that large errors in one of the modes as with m = 1 above do not jeopardize the overall accuracy of the approximation), but not so many that a large number of modes do not satisfy (22) well, we may use (22) to obtain a modified version of (13):

$$E(\rho,z_{11}/2) \simeq \left(\frac{b-\rho}{\pi\rho}\right)^{\frac{1}{2}} e^{-ik_oz_{11}-7\pi i/8}[E_o(b-\rho)+E_1(b-\rho)] \quad (23)$$

In Figs. 3-5, we compare the results of (23) with those of the exact mode series for several different beam widths. Except near $\rho = 0$ where (23) is still nonuniform, we see that (23) provides a very good approximation indeed. The fact that this waveguide takes a field concentrated near the center of the guide at $z=0$ and transforms it into a "ring" near the outer boundary at $z=z_{11}/2$ may make this Fresnel image useful in some types of beam transformation devices.

Conclusion

We have attempted to extend this method to other possible Fresnel image planes, such as $z_{11}/4$, $z_{11}/3$, and so on. This has met with only very limited success, due to the increased number of values of ρ where the approximations become nonuniform, as they do at $\rho=0$ and $\rho=b$ in eqn.(8). What seems to be needed is a uniform approximation which, though more complicated, would enable us to accurately predict the form of an image at any Fresnel image plane.

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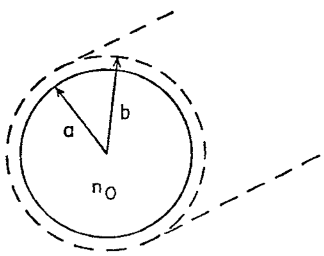

Fig. 1: Step-index fiber and equivalent hollow metallic guide.

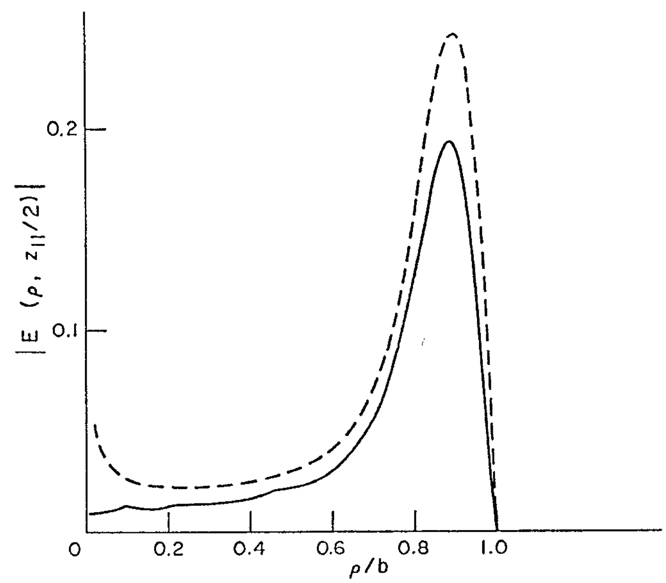

Fig. 2: Exact (——) and approximate (---; eqn.(13)) Fresnel image at $z = z_{11}/2$; $\alpha = w_0/b = 0.1$.

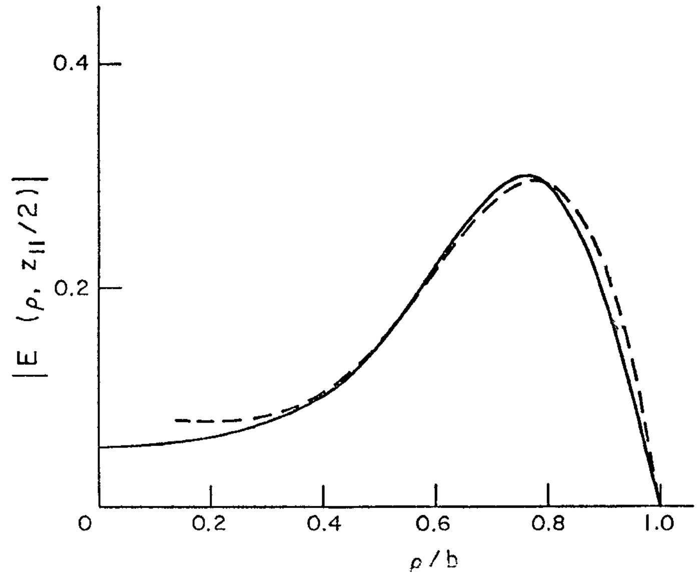

Fig. 3: Exact (——) and approximate (---; eqn.(23)) Fresnel image at $z = z_{11}/2$; $\alpha = w_0/b = 0.2$.

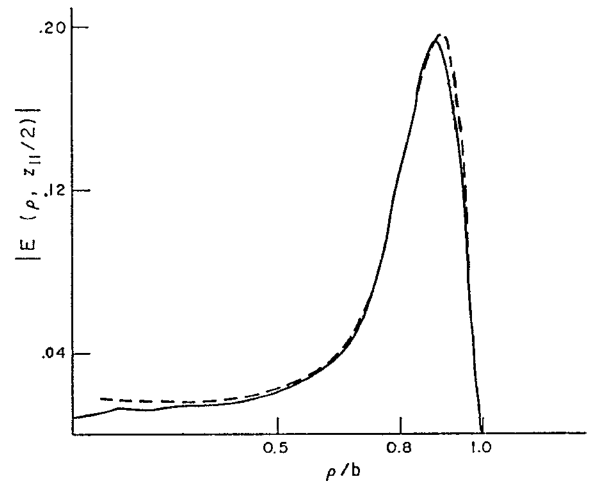

Fig. 4: Same as Fig. 3, but $w_0/b = 0$.

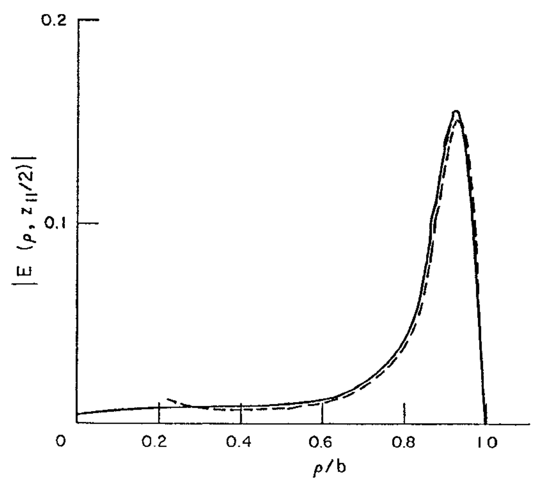

Fig. 5: Same as Fig. 3 but $w_0/b = 0.067$.